double spectral functions implies the following reflection relationship:

$$F^*(l^*,s^*)=-F(l,s)\exp(-2\pi il). \tag{4.8}$$

We now define the generalized S function to be

$$S(l,s)=1+2iF(l,s)e^{-2\pi il}. \tag{4.9}$$

Its continuation to the first unphysical sheet satisfies the property

$$S_u(l,s-)=S(l,s+). \tag{4.10}$$

From (4.7) and (4.8) we thus obtain

$$S_u(l,s)=(1-e^{-2\pi il})+S^{-1}(l,s)e^{-2\pi il}. \tag{4.11}$$

Clearly, when l is an integer, we regain (2.3). When l is not an integer, the definition of $S(l,s)$ has made possible the association of a pole in the unphysical sheet with a zero in $S(l,s)$. It therefore follows that the logarithmic function which we should be interested in is

$$K(l,s)=\ln S(l,s). \tag{4.12}$$

To eliminate the elastic cut for nonintegral l, it is the dispersion relation for $K(l,s)/(s-s_1)^{l+\frac{1}{2}}$ which we must consider. The remarks at the end of Sec. II are therefore especially relevant in the use of the iteration method.

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Representations and Mass Formulas for SU(4)

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A hierarchial scheme of SU(n) symmetries among the strong interactions is proposed with the $n-1$ additive quantum numbers exactly conserved. Simple dynamics as well as mass-splitting considerations are shown to favor the assignment of the mesons to the 15-dimensional representations and the baryons and $\frac{3}{2}^+$ isobars to two inequivalent 20-dimensional representations of SU(4). The predictions of new particles are discussed.

THE experimental studies of the boson mass spectrum in the region 400–1600 MeV[1] suggest a structure far too complicated for a simple SU(3) model to cope with. It seems worthwhile therefore to look for supersymmetries that would have SU(3) as a subgroup and that would also have large representations suitable for containing, e.g., all the known or suggested vector mesons. SU(4) suggests itself as the most obvious candidate for such a supersymmetry.[2] In this scheme it is also easy to formulate a baryon-lepton symmetry[3] of the Cabibbo type.[4]

In an earlier paper[2] it was suggested that SU(4) would be physically relevant only for the vector mesons, the reason being their property of bootstrapping themselves. For other multiplets the breakdown of the symmetry would be catastrophic and the conservation of the new additive quantum number Z, supercharge (also called hyperstrangeness), would lose all meaning.

In this paper we adopt the point of view of an exactly conserved supercharge. The mesons, both pseudoscalar (M) and vector (V), belong to the regular (adjoint) 15-dimensional representation $\psi_i\bar{\psi}_j-\delta_{ij}$, where ψ_i ($i=1\cdots4$) is the basic four-component quark field of SU(4). The baryons (B) are included in $\psi\psi\bar{\psi}$ and the $\frac{3}{2}^+$ isobars (B^*) in some representation of the baryon-meson system. We characterize the irreducible representations by the combination $(\lambda\mu\nu)$ in analogy to the common SU(3) usage. In Table I we give the dimensionalities

$$d=(\lambda+1)(\mu+1)(\nu+1)(\lambda+\mu+2)\times(\mu+\nu+2)(\lambda+\mu+\nu+3)/12$$

and SU(3)-multiplet contents of some of the representations. Notice that there are three inequivalent 20-dimensional representations. The most interesting

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TABLE I. Dimensionalities d and SU(3) contents of some SU(4) representations ($\lambda\mu\nu$).

$\lambda\mu\nu$	d	SU(3) content
000	1	1
100	4	3; 1
010	6	$\bar{3}$; 3
200	10	6; 3; 1
020	20′	$\bar{6}$; 8; 6
110	20	8; 6, $\bar{3}$; 3
101	15	3; 8, 1; $\bar{3}$
300	20″	10; 6; 3; 1
030	50	$\overline{10}$; $\overline{15}$; 15; 10
210	45	15; 10, 8; 6, $\bar{3}$; 3
201	36	6; 15, 3; 8, 1; $\bar{3}$
120	60	$\overline{15}$; 15, $\bar{6}$; 10, 8; 6
111	64	8; $\overline{15}$, 6, $\bar{3}$; 15, $\bar{6}$, 3; 8
202	84	6; 15, 3; 27, 8, 1; $\overline{15}$, $\bar{3}$; $\bar{6}$
211	140	15; 27, 10, 8; 24, $\overline{15}$, 6, $\bar{3}$; 15, $\bar{6}$, 3; 8
121	175	$\overline{15}$; $\overline{24}$, 15, $\bar{6}$; 27, 10, $\overline{10}$, 8; 24, $\overline{15}$, 6; 15
303	300	10; 24, 6; 42, 15, 3; 64, 27, 8, 1; $\overline{42}$, $\overline{15}$, $\bar{3}$; $\overline{24}$, $\bar{6}$; $\overline{10}$

reduction formulas for products of representations will be

$$4\times\bar{4}=15+1\,,$$
$$4\times 15=36+\overline{20}+4\,,$$
$$15\times 15=84+45+\overline{45}+20'+15+15+1\,,$$
$$15\times 20=140+\overline{60}+\overline{36}+20+20+20''+\bar{4}\,,$$
$$20\times\overline{20}=175+84+45+\overline{45}+20'+15+15+1\,,$$

and

$$20''\times\overline{20}''=300+84+15+1\,.$$

In order to see whether the baryons belong to the 36-dimensional or the 20-dimensional representation, both of which contain a suitable SU(3)-octuplet, we have calculated the static Born force for 4×15 "B"-M

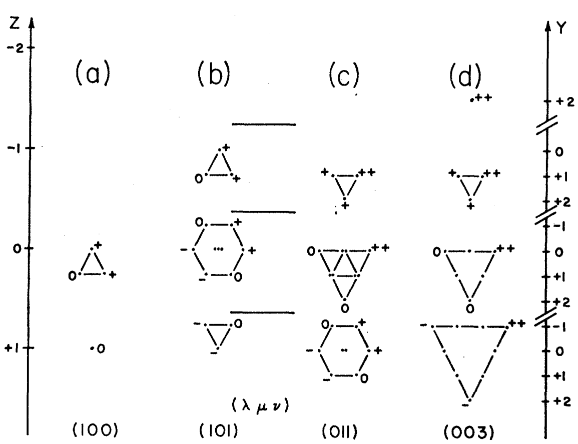

FIG. 1. (a) shows the quantum number structure of the fundamental (100) representation the signs standing for the charges. (b), (c), and (d) illustrate the suggested meson, baryon, and isobar supermultiplets, respectively. $-Y$ = hypercharge, Z = supercharge.

TABLE II. Splitting modes for the SU(4) multiplet (110).

Type of splitting	SU(3) multiplet 8	6	$\bar{3}$	3
1	1	1	1	1
15	3	1	-3	-7
15′	3	-6	4	0
84	-3	2	12	-8

scattering using a method described in previous work on SU(3).[5,6] The bilinear Casimir operator of SU(4), which is needed in this calculation, has the value

$$C'=3\lambda^2+4\mu^2+3\nu^2+4\lambda\mu+2\lambda\nu+4\mu\nu+12\lambda+16\mu+12\nu$$

for the representation ($\lambda\mu\nu$).

The results show that there is attraction in the 20-dimensional, but repulsion in the two other representations. We thus assign the baryons to a supermultiplet whose structure is shown in Fig. 1(c). Figures 1(a) and 1(b) show the quark and meson representations.

Turning now to the isobars we notice that a suitable SU(3)-decuplet occurs in the representations 140 and 20″ of the reduction 15×20. Because the representation 20 occurs twice in this reduction, the meson baryon coupling involves a free and unknown mixing parameter analogous to F/D mixing in the SU(3) scheme. The forces unfortunately depend on the value of this parameter. We are thus forced to assign the isobars to the representation 20″ [Fig. 1(d)] only on grounds of simplicity and the fact that it is inconceivable that the decuplet would be much lower in energy than the remaining 130 states, especially as 35 of them have the same supercharge as the decuplet.

We now want to discuss the breakdown of SU(4). There are many ways to study modes of mass splittings and to obtain mass formulas. We will assume that there is a hierarchial scheme of broken SU(n) symmetries and that SU(4) is broken mainly in a way leaving the SU(3) multiplets of constant supercharge degenerate.

Tables II–IV give the possible splitting modes for the baryons, isobars, and mesons, respectively. The tensorial transformation properties of the modes are characterized by the dimensionality of the corresponding SU(4) multiplet.[7,8]

TABLE III. Splitting modes for the SU(4) multiplet (300).

Type of splitting	SU(3) multiplet 10	6	3	1
1	1	1	1	1
15	-3	1	5	9
84	3	-7	-2	18
300	1	-5	10	-10

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TABLE IV. Splitting modes for the SU(4) multiplet (101).

Type of splitting	SU(3) multiplet 8	3, $\bar{3}$	1
1	1	1	1
15	1	−1	−2
84	−1	4	−16

From Table III we see that the only splitting mode for which the SU(3) decuplet can possibly lie lowest in energy is the 15-dimensional one. This corresponds to a perturbing term in the Hamiltonian transforming like the 15th (singlet) component of the regular representation. This is in exact analogy with the Gell-Mann–Okubo[9] type of splitting in the case of SU(3) breakdown. A self-consistency argument similar to the one used previously for[7] SU(3) allows us to treat only "pure" splitting modes, i.e., even a small component of 84-splitting would destroy the stability of the nondegenerate solutions.

For the isobars we thus get an equal spacing rule in the Z direction. For the baryons there are two equivalent 15-dimensional splitting modes that can mix. Again the remaining mode (84) cannot lead to the observed situation, where the SU(3) octuplet lies lowest. We can thus write the splitting as

$$\cos\theta(15)+\sin\theta(15'),$$

where the mixing parameter θ goes from $-90°$ to $90°$. If we normalize the splittings as in Table II we have the result that the octuplet lies lowest when $-2/9<\tan\theta<6$. The next lowest multiplets in this region are the sextuplet $(Z=0)$ for $\tan\theta<\frac{2}{5}$ and the triplet $(Z=0)$ for $\tan\theta>\frac{2}{5}$.

Our choice of quantum numbers for the basic fields was suggested by the baryon-lepton symmetry. We now see that it leads to a scheme where no new nonstrange baryonic states are predicted. The most interesting predictions are thus the $T=1$ and $T=0$ states in the $Y=0$ channel. It would be nice to find them below 2000 MeV, so that SU(4) would not be too badly broken.

For the vector mesons we get from Table IV the mass formula

$$3m_6{}^2=m_8{}^2+2m_1{}^2$$

for the 15-dimensional splitting mode. The subscripts denote the SU(3) dimensionalities. Inserting $m_8=820$ MeV as a mean value for the octuplet and $m_1=1020$ MeV for the φ meson we get $m_6\approx955$ MeV. If one uses inverse squares of the masses[10] the result is $m_6\approx940$ MeV. The fact that $\varphi\rightarrow K\bar{K}$ is forbidden in SU(4), if φ is a pure SU(3) singlet, requires a considerable $\varphi-\omega$ mixing. The effect of this mixing would be to lower the mass m_6 somewhat. The original inspiration for SU(4) studies came from the κ resonance (725 MeV). Whether or not this resonance exists, it would be interesting to study in detail the mass region 850–1000 MeV in order to find the predicted supercharged vector mesons.

The pseudoscalar mesons do not show an SU(3) degeneracy as striking as the one for the other multiplets discussed above. This leads us to conjecture that the seven particles missing from the regular representation of SU(4) have a relatively large mass, say around 1000 MeV. This would then be the reason why the pseudoscalar counterpart of the φ has not been established. One possible candidate is the 960-MeV $\eta\pi\pi$ resonance.[11] The particles with nonzero supercharge can of course be produced only in associated production here as well as in the vector meson case.

Finally, we give the analog of the Gell-Mann–Okubo mass formula for SU(4). A form valid for the relevant representations is easily derived from Tables II–IV and reads

$$m=a+bZ+c(Z^2-C),$$

where $C=\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu$ is the Casimir operator for the SU(3) representation.[12] For the isobars, Z^2-C is of the form $a'+b'Z$, so that the mass formula degenerates to $m=a'+b'Z$. For the mesons $b=0$ and m^2 (or m^{-2} for V) should be used instead of m.

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[12] This is a special case of the mass formula for general SU(n) given by M. L. Whippman, University of Pennsylvania (to be published).